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$\mathcal{N}=2$ SO(N) SYM theory from matrix model

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Abstract: We study $\mathcal{N}=2$ SO($2N+1$) SYM theory in the context of matrix model. By adding a superpotential of the scalar multiplet, $W(\Phi)$, of degree $2N+2$, we reduce the theory to $\mathcal{N}=1$. The $2N+1$ distinct critical points of $W(\Phi)$ allow us to choose a vacuum in such a way to break the gauge group to its maximal abelian subgroup. We compute the free energy of the corresponding matrix model in the planar limit and up to two vertices. This result is then used to work out the effective superpotential of $\mathcal{N}=1$ theory up to one-instanton correction. At the final step, by scaling the superpotential to zero, the effective U(1) couplings and the prepotential of the $\mathcal{N}=2$ theory are calculated which agree with the previous results.

Keywords: Nonperturbative Effects, Supersymmetry and Duality, Supersymmetric Effective Theories.

Contents

1. Introduction

The study of $\mathcal{N} = 1$ supersymmetric gauge theories has proven important in understanding the more realistic theories such as QCD. This is because on one hand they share many common properties like chiral symmetry breaking, the existence of a mass gap, and color confinement in the infrared. And on the other hand, supersymmetry puts strict, though tractable, conditions on the dynamics of the theory which makes the theory easier to analyze. Therefore, a thorough understanding of supersymmetric gauge theories will help in unraveling the low energy phenomena, of the kind mentioned above, of the corresponding nonsupersymmetric theories. This is one, among many others, main reason that supersymmetric gauge theories are so appealing to study.

A remarkable advance in the understanding of supersymmetric gauge theories and their relations to Matrix models has recently been achieved through the work of Dijkgraaf and Vafa [1]–[4]. Consider $\mathcal{N} = 2$ supersymmetric lagrangian which consists of an $\mathcal{N} = 2$ vector multiplet (A, Φ) in the adjoint representation of the gauge group $\mathrm{U}(N)$. Here A and Φ are $\mathcal{N} = 1$ vector and chiral multiplets respectively. Upon adding a superpotential $W(\Phi)$ to the $\mathcal{N} = 2$ lagrangian, the supersymmetry gets reduced to $\mathcal{N} = 1$. Dijkgraaf and Vafa have put forward the proposal that the low energy dynamics of this $\mathcal{N} = 1$ theory can be completely determined by perturbative calculations of the free energy of a zero dimensional matrix model in the planar limit. The potential of the matrix model is taken to be the same as

$W(\Phi)$, but with Φ's regarded as constant $M\times M$ matrices in the Lie algebra of $\mathrm{U}(M)$. The most important feature of this correspondence is that by perturbative calculations in the matrix model side one learns about the nonperturbative effects — mainly due to instantons — in the gauge theory side. Specifically, let $W(\Phi)$ be a polynomial of degree $n+1$ in Φ. The classical supersymmetric vacuum is then characterized by a constant diagonal matrix with elements e_i, the critical points of $W(\Phi)$. Let N_i indicate the multiplicity of e_i in the vacuum such that $N=\sum_i^n N_i$. This choice of vacuum breaks the gauge symmetry as follows

$$\mathrm{U}(N)\to \mathrm{U}(N_1)\times \mathrm{U}(N_2)\times\cdots\times \mathrm{U}(N_n)\,.$$

The instantons contributions to the effective superpotential are then given by

$$W_{\rm eff}^{\rm inst}=-\sum_i N_i\frac{\partial \mathcal{F}_0}{\partial S_i}\,,$$

where $\mathcal{F}_0$ is the free energy of the matrix model in the planar limit, and $S_i=g_sM_i$.

Using the perturbative calculations in the matrix models, the effective superpotential of a wide class of $\mathcal{N}=1$ supersymmetric gauge theories has been obtained in complete agreement with the earlier results. Interestingly, one can go even one step further to extract information about the low energy dynamics of the $\mathcal{N}=2$ theory itself. This can be done as follows. One introduces a superpotential $\alpha W(\Phi)$ of degree $N+1$, with α a real parameter, breaking the $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. Since $W(\Phi)$ has N critical points, one can choose the vacuum as

$$\Phi_0=\mathrm{diag}(e_1,e_2,\ldots,e_N)\,,$$

therefore the $\mathrm{U}(N)$ gauge group classically breaks to $\mathrm{U}(1)^N$. By adding the superpotential $W(\Phi)$, one in fact freezes the whole classical vacuum manifold C^N of $\mathcal{N}=2$ theory to a point Φ_0, the vacuum of $\mathcal{N}=1$ theory. In conclusion, one computes the effective superpotential of this theory and notices that there are some low energy quantities which are independent of the parameter α, and hence must be identified with the corresponding quantities in the $\mathcal{N}=2$ theory. In this way, using the perturbative analysis of the matrix model, the prepotential of $\mathcal{N}=2$ $\mathrm{U}(2)$ theory was rederived in [5]. This method was further generalized for the gauge group $\mathrm{U}(N)$, and again with complete agreement with the Seiberg-Witten solution of $\mathcal{N}=2$ $\mathrm{U}(N)$ gauge theory [6]. It is our aim in this paper to work out the Seiberg-Witten solution of $\mathcal{N}=2$ $\mathrm{SO}(N)$ gauge theory by perturbative computations of the free energy of the corresponding matrix model.

In the above context of gauge theory/matrix model correspondence, $\mathcal{N}=1$ SO / SP gauge theories have also been examined from different points of views [7]–[13]. The perturbative matrix model language, though, has only been used to analyze the gauge theory in the trivial vacuum sector. To derive the $\mathcal{N}=2$ results, as mentioned above, we need to choose a vacuum which breaks $\mathrm{SO}(2N)$ or $\mathrm{SO}(2N+1)$ gauge group to $\mathrm{U}(1)^N$ representing a typical point on the Coulomb branch of $\mathcal{N}=2$ vacuum moduli space, and then performing the perturbative calculations around the corresponding matrix model vacuum. This is what we will do in the next section.

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The organization of this paper is as follows. In section 2, we introduce the matrix model action including the fluctuations around the vacuum and their ghosts counterparts which are necessary for our special gauge fixing. In section 3, we calculate the free energy of the matrix model which consists of three parts; 1) the nonperturbative part including the contribution of the group volume and quadratic integrals, 2) two loop planar free energy and 3) unoriented planar graphs. In section 4, we derive the effective action, and show that the coupling constants τ_{ij} can be obtained from the free energy of the matrix model by a variant form of the Vafa-Dijkgraaf prescription. The result is then reexpressed in terms of the periods a_i's. We conclude in section 5 and derive the Vandermonde determinant of the Fadeev-Popov ghosts in the appendix.

2. The Matrix Model superpotential

In this section, we introduce a superpotential $W(\Phi)$ of the scalar multiplet which is of degree $2N+2$ for the group $\mathrm{SO}(2N+1)$. $W(\Phi)$ is chosen such that it has $2N+1$ distinct critical points e_i's. To be explicit, let us introduce the superpotential as

$$W(\Phi) = \alpha \sum_{l=0}^{N} \frac{s_{N-l}(e^2)}{2l+2} \operatorname{tr} \Phi^{2l+2}, \tag{2.1}$$

where

$$s_m(e^2) = \sum_{i_1<i_2<\cdots<i_m} e_{i_1}^2 e_{i_2}^2 \cdots e_{i_m}^2, \tag{2.2}$$

and α is a real parameter which, at the end, is scaled to zero to read off the effective $\mathrm{U}(1)$ gauge couplings of the $\mathcal{N}=2$ effective theory. e_i's are the critical points of $W(x)$

$$W'(x) = \alpha x \sum_{l=0}^{N} s_{N-l}(e^2) x^{2l} = \alpha x \prod_{i=1}^{N} (x^2 + e_i^2) \equiv \alpha w'(x). \tag{2.3}$$

Taking the vacuum as

$$\Phi_0 = \mathrm{diag}(0,\ e_1 i\sigma_2,\ e_2 i\sigma_2, \ldots,\ e_N i\sigma_2), \tag{2.4}$$

will break the gauge group classically as

$$\mathrm{SO}(2N+1) \to \mathrm{U}(1)^N. \tag{2.5}$$

In the matrix model side, as mentioned before, one takes the same $W(\Phi)$ playing the role of the potential of the model, but with Φ's now considered as constant $2M \times 2M$ matrices in the Lie algebra of $\mathrm{SO}(2M)$.[1] To set up the perturbation theory, let us expand the superpotential around the critical points of $W(\Phi)$. This we do by substituting $\Phi \to \Phi_0 + \Psi$ in $W(\Phi)$, where Φ_0 is the vacuum

$$\Phi_0 = \mathrm{diag}(\mathbf{0}_{2M_0\times 2M_0},\ e_1 i\sigma_2 \otimes \mathbf{1}_{M_1\times M_1},\ e_2 i\sigma_2 \otimes \mathbf{1}_{M_2\times M_2},\ \ldots,\ e_N i\sigma_2 \otimes \mathbf{1}_{M_N\times M_N}). \tag{2.6}$$

[1]Note that the size of the matrices M_i in the matrix model are chosen to be large by multiplying N_i by a large number k_i. For convenience, we have taken k_i's to be even so that the matrix model we obtain is even dimensional.

This choice of vacuum will break the gauge group of the matrix model as follows

$$\mathrm{SO}(2M) \to \mathrm{SO}(2M_0) \times \mathrm{U}(M_1) \times \mathrm{U}(M_2) \times \cdots \times \mathrm{U}(M_N)\,, \tag{2.7}$$

so that

$$M = \sum_{i=0}^{N} M_i\,. \tag{2.8}$$

Upon considering the small fluctuations around the vacuum (2.6), up to the second order in Ψ, $W(\Phi)$ reads

$$W(\Phi) = \sum_{i=0}^{N} M_i W(\imath e_i) + \frac{1}{2}\alpha \sum_{l=0}^{N} s_{N-l} \sum_{m=0}^{l} \mathrm{Tr}(\Psi \Phi_0^m \Psi \Phi_0^{2l-m}) + \mathcal{O}(\Psi^3)\,. \tag{2.9}$$

Further, it is easy to show that the quadratic part is as follows,

$$W_2 = \frac{1}{2}\alpha s_N \,\mathrm{Tr}(\psi_{00}\psi_{00}) + \frac{1}{2}\alpha \sum_{i=1}^{N}\sum_{l=0}^{N} l s_{N-l}(\imath e_i)^{2l} 2\,\mathrm{Tr}(\psi_{ii}^0\psi_{ii}^0 - \psi_{ii}^2\psi_{ii}^2)\,, \tag{2.10}$$

where we have decomposed the $2M_i \times 2M_j$ ψ_{ij} matrices in terms of $\sigma_\mu\,, \mu = 0,1,2,3$ matrices, with σ_i the Pauli matrices and $\sigma_0 \equiv \mathbf{1}_{2\times 2}$,

$$\psi_{ij} \equiv \psi_{ij}^0 \otimes \sigma_0 + \psi_{ij}^1 \otimes \sigma_1 + \imath \psi_{ij}^2 \otimes \sigma_2 + \psi_{ij}^3 \otimes \sigma_3\,, \tag{2.11}$$

ψ_{ij}^μ are now $M_i \times M_j$ matrices. The important point to notice here is that there are elements of Ψ which are absent in the quadratic part of the action. These include ψ_{ij} for $i \neq j$ and ψ_{ii}^β for $\beta = 1,3$. Therefore, these are not propagating fields and one should gauge them away. Note that the total number of degrees of freedom that we are going to gauge away is exactly equal to the number of broken gauge generators in (2.7), i.e.,

$$4M_0 \sum_{i=1}^{N} M_i + 4\sum_{i<j}^{N} M_i M_j + \sum_{i=1}^{N} M_i(M_i - 1)\,. \tag{2.12}$$

As we will show in the appendix, the gauge fixing can be implemented by introducing the Faddeev-Popov ghosts. The ghost action takes the following form

$$\frac{1}{4}\,\mathrm{Tr}\, B[\Phi, C] = \frac{1}{4}\,\mathrm{Tr}\, B[\Phi_0, C] + \frac{1}{4}\,\mathrm{Tr}\, B[\Psi, C]\,. \tag{2.13}$$

The kinetic part of the ghost action can be obtained by expanding the ghost action around the vacuum

$$\begin{aligned}\frac{1}{4}\,\mathrm{Tr}\, B[\Phi_0, C] = &- \sum_i \mathrm{Tr}\left[(B_{i0}^1 C_{0i}^3 - B_{i0}^3 C_{0i}^1) - (B_{i0}^0 C_{0i}^2 + B_{i0}^2 C_{0i}^0)\right] e_i - \\ &- \sum_i 2\,\mathrm{Tr}\left[B_{ii}^1 C_{ii}^3 - B_{ii}^3 C_{ii}^1\right] e_i - \sum_{i<j} \mathrm{Tr}\left[B_{ji}^1 C_{ij}^3 - B_{ji}^3 C_{ij}^1\right](e_i + e_j) - \\ &- \sum_{i<j} \mathrm{Tr}\left[B_{ji}^0 C_{ij}^2 + B_{ji}^2 C_{ij}^0\right](e_i - e_j)\,.\end{aligned} \tag{2.14}$$

Let us then fix the gauge to $\psi_{ij} = 0$ for $i \neq j$, and $\psi^{\beta}_{ii} = 0$ for $\beta = 1, 3$. Doing so, the interacting part of the ghost action becomes,

$$
\begin{aligned}
\frac{1}{4}\operatorname{Tr} B[\Psi, C] = \sum_{i<j} \operatorname{Tr}\Big[& (B^{\mu}_{ji}\psi^{0}_{ii}C^{\mu}_{ij} - B^{\mu}_{ji}C^{\mu}_{ij}\psi^{0}_{jj}) - \\
& - (B^{1}_{ji}\psi^{2}_{ii}C^{3}_{ij} + B^{1}_{ji}C^{3}_{ij}\psi^{2}_{jj}) + (B^{3}_{ji}\psi^{2}_{ii}C^{1}_{ij} + B^{3}_{ji}C^{1}_{ij}\psi^{2}_{jj}) - \\
& - (B^{2}_{ji}\psi^{2}_{ii}C^{0}_{ij} - B^{2}_{ji}C^{0}_{ij}\psi^{2}_{jj}) - (B^{0}_{ji}\psi^{2}_{ii}C^{2}_{ij} - B^{0}_{ji}C^{2}_{ij}\psi^{2}_{jj})\Big] + \\
& +2\sum_{i} \operatorname{Tr}\left[(B^{3}_{ii}\psi^{2}_{ii}C^{1}_{ii} - B^{1}_{ii}\psi^{2}_{ii}C^{3}_{ii}) + (B^{1}_{ii}\psi^{0}_{ii}C^{1}_{ii} + B^{3}_{ii}\psi^{0}_{ii}C^{3}_{ii})\right] + \\
& + \sum_{i} \operatorname{Tr}\left[(B^{3}_{i0}C^{1}_{0i}\psi^{2}_{ii} - B^{1}_{i0}C^{3}_{0i}\psi^{2}_{ii}) + (B^{2}_{i0}C^{0}_{0i}\psi^{2}_{ii} + B^{0}_{i0}C^{2}_{0i}\psi^{2}_{ii})\right] - \\
& - \sum_{i} \operatorname{Tr}(B^{\mu}_{i0}C^{\mu}_{0i}\psi^{0}_{ii}) + \frac{1}{2}\sum_{i}\operatorname{Tr}(B_{i0}\psi_{00}C_{0i})\,. \qquad (2.15)
\end{aligned}
$$

The kinetic and interaction parts of the "bosonic" action, on the other hand, are found in this gauge to be

$$
\begin{aligned}
W_2 &= \frac{1}{2}\alpha s_N \operatorname{Tr}(\psi_{00}\psi_{00}) + \frac{1}{2}\alpha\sum_{i=1}^{N}\sum_{l=0}^{N}(\imath e_i)^{2l} l s_{N-l} 2\operatorname{Tr}(\psi^{0}_{ii}\psi^{0}_{ii} - \psi^{2}_{ii}\psi^{2}_{ii}) \\
W_3 &= -\alpha\sum_{i=1}^{N}\sum_{l=0}^{N} 2l(2l+1)(\imath e_i)^{2l-1}(-\imath)s_{N-l}\operatorname{Tr}(\psi^{222}_{ii} - 3\psi^{200}_{ii}) \\
W_4 &= -\alpha\frac{s_{N-1}}{4}\operatorname{Tr}(\psi_{00})^4 - \alpha\sum_{i=1}^{N}\sum_{l=0}^{N}\frac{1}{6}(2l-1)2l(2l+1)(\imath e_i)^{2l-2}s_{N-l}\times \\
&\quad \times 2\operatorname{Tr}(\psi^{2222}_{ii} - 4\psi^{2200}_{ii} - 2\psi^{2020}_{ii} + \psi^{0000}_{ii})\,, \qquad (2.16)
\end{aligned}
$$

where we have used the notation $\psi^{ab\ldots c} = \psi^a\psi^b\ldots\psi^c$ for $a, b, \ldots = 0, 2$. Performing the sum over l we obtain

$$
\begin{aligned}
W_2 &= \frac{1}{2}\alpha\Delta_0\operatorname{Tr}(\psi_{00}\psi_{00}) - \frac{1}{2}\alpha\sum_{i=1}^{N}\Delta_i\operatorname{Tr}(\psi^{0}_{ii}\psi^{0}_{ii} - \psi^{2}_{ii}\psi^{2}_{ii})\,, \\
W_3 &= -\imath\alpha\sum_{i=1}^{N}\frac{\gamma_{3,i}}{3}\operatorname{Tr}(\psi^{222}_{ii} - 3\psi^{200}_{ii})\,, \\
W_4 &= -\alpha\frac{s_{N-1}}{4}\operatorname{Tr}(\psi_{00})^4 + \alpha\sum_{i=1}^{N}\frac{\gamma_{4,i}}{4}\operatorname{Tr}(\psi^{2222}_{ii} - 4\psi^{2200}_{ii} - 2\psi^{2020}_{ii} + \psi^{0000}_{ii})\,, \qquad (2.17)
\end{aligned}
$$

where use has been made of

$$
\sum_{l=0}^{N} l s_{N-l}(\imath e_i)^{2l} = \frac{\imath e_i}{2}\left[\frac{d}{dx}\prod_{k=1}^{N}(x^2 + e_k^2)\right]_{x=\imath e_i} = -e_i^2\prod_{k\neq i}\left(e_k^2 - e_i^2\right)\,, \qquad (2.18)
$$

together with the following definitions

$$
\begin{aligned}
\Delta_i &\equiv 2e_i^2 R_i\,, \qquad \Delta_0 \equiv s_N \\
\gamma_{3,i} &\equiv \imath e_i R_i \left(3 + 4e_i^2 \sum_{j\neq i} \frac{1}{e_{ij}} \right) \\
\gamma_{4,i} &\equiv R_i \left(1 + 8e_i^2 \sum_{j\neq i} \frac{1}{e_{ij}} + 4e_i^4 \sum_{m\neq i} \sum_{n\neq i,m} \frac{1}{e_{im}e_{in}} \right) \\
R_i &\equiv \prod_{k\neq i} (e_k^2 - e_i^2)\,, \qquad e_{ij} \equiv e_i^2 - e_j^2\,.
\end{aligned} \tag{2.19}
$$

The higher interaction vertices are given by

$$
\gamma_{p,i} = \frac{1}{(p-1)!} \left[\left(\frac{\partial}{\partial x} \right)^{p-1} x \prod_{j=1}^{N} (x^2 + e_j^2) \right]_{x=\imath e_i} . \tag{2.20}
$$

With the matrix model perturbative action in hand, now we can find the free energy of the matrix model by which the gauge theory effective action and other related quantities are found in the following sections.

3. Matrix model free energy

In this section we calculate the free energy $\mathcal{F}_0$ of the matrix model which consists of two parts;

1. The non-perturbative part which comes from the volume of the gauge group and the integration over the quadratic part of the action.

2. The perturbative parts which are coming from the interacting parts of the matrix model.

Having obtained the free energy $\mathcal{F}_0$, we use the prescription given by Dijkgraaf and Vafa [1] to write down the $\mathcal{N}=1$ effective superpotential $W_{\text{eff}}(S)$, which is

$$
W_{\text{eff}}(S) = -\sum_{i=1}^{N} \left(\frac{\partial \mathcal{F}_0}{\partial S_i} + \lambda \mathcal{G}_0 - 2\pi i \tau S_i \right), \tag{3.1}
$$

where $\mathcal{G}_0$ is the contributions of the unoriented planar graphs to the free energy, and $\lambda = 4$ for the SO(N) group [7]. τ is the bare coupling, and we have set $N_i = 1$. Moreover, as in the case of U(N), the effective U(1) couplings can in principle be calculated through the formula

$$
2\pi\imath\tau_{ij}(e) = \left(\frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \right)_{\langle S_i \rangle} , \tag{3.2}
$$

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where $\langle S_i \rangle$ are the vev of the gluinos obtained by extremizing the effective superpotential $W_{\rm eff}(S)$. However, the case of $\mathrm{SO}(N)$ group is a bit subtle and formula (3.2) needs modification.[2] The reason for this is as follows. In the double line notation of 't Hooft, two index lines of the antisymmetric representations of $\mathrm{SO}(N)$ group have the same orientation (as opposed to the case of adjoint representation of $\mathrm{U}(N)$ group). On the field theory side, since gauge fields W^α act through the commutator on matter adjoint fields — antisymmetric representations of $\mathrm{SO}(N)$ — one gets an extra minus sign when one moves one of the W^α to the outer index loop. These are the graphs contributing to the effective U(1) couplings. In order to take into account this extra minus sign, in each loop diagram of (anti)symmetric field we assign an S_i to one index loop and an $\widetilde{S}_j$ to the adjacent index loop. All this amounts to modifying (3.2) to

$$2\pi\imath\tau_{ij} = \left(\frac{\partial^2 \mathcal{F}_0(S,\widetilde{S})}{\partial S_i \partial S_j}\right)_{\langle S_i \rangle}, \tag{3.3}$$

noticing that

$$\frac{\partial \widetilde{S}_i}{\partial S_j} = -\delta_{ij}\,. \tag{3.4}$$

And after differentiation setting $\widetilde{S}_i = S_i$.

Knowing the effective U(1) couplings we can proceed to calculate the prepotential of $\mathcal{N}=2$ theory. Recall that the $\mathcal{N}=2$ prepotential is expressed in terms of the periods a_i's. Therefore, if we reexpress (3.3) in terms of a_i's, we can work out the $\mathcal{N}=2$ prepotential $\mathcal{F}(a)$ by a double integration of the following formula

$$\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j}\,. \tag{3.5}$$

3.1 Nonperturbative part of the free energy

The nonperturbative part of the matrix model free energy $\mathcal{F}_0^{(\rm np)}$comprises of three parts. These include the integral over the kinetic terms of ψ_{ii}'s, those of ghosts B_{ji}, C_{ij}, and the volume factor of the broken gauge group. Let us discuss each part separately with some detail. First, the kinetic terms of ψ's consist of three parts:

$$W_{\rm kin}(\psi) = \frac{\alpha}{2}\left(-\Delta_0 \mathrm{Tr}(\psi_{00})^2 - \sum_{\rm i=1}^{\rm N} \Delta_{\rm i}\,\mathrm{Tr}(\psi^0_{\rm ii})^2 + \sum_{\rm i=1}^{\rm N} \Delta_{\rm i}\,\mathrm{Tr}(\psi^2_{\rm ii})^2\right). \tag{3.6}$$

Accordingly, the gaussian integral over ψ's can be performed easily, giving the result

$$\begin{aligned}\int d\psi \exp\left(-\frac{1}{g_s}W_{kin}(\psi)\right) &= \left(\frac{2\pi g_s}{2\alpha\Delta_0}\right)^{\frac{1}{2}M_0(2M_0-1)} \times \\ &\times \prod_i \left\{ \left(\frac{2\pi g_s}{2\alpha\Delta_i}\right)^{\frac{1}{4}M_i(M_i-1)} \left(\frac{2\pi g_s}{2\alpha\Delta_i}\right)^{\frac{1}{4}M_i(M_i-1)} \left(\frac{2\pi g_s}{\alpha\Delta_i}\right)^{\frac{1}{2}M_i} \right\}.\end{aligned}$$

[2]This is also noticed in [15] for the case of $\mathrm{U}(N)$ group with matter in (anti)symmetric representation.

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Taking into account the appropriate g_s factors, and ignoring the linear terms in M_0, M_i in the planar limit, gives rise to a contribution to $\mathcal{F}_0^{(\mathrm{np})}(S)$ of the form

$$S_0^2 \log\left(\frac{\pi g_s}{\alpha\Delta_0}\right) + \frac{1}{2}\sum_i S_i^2 \log\left(\frac{\pi g_s}{\alpha\Delta_i}\right). \tag{3.7}$$

Now, we consider the ghost sector. There are three types of ghosts B, C which correspond to the blocks $(ii), (i0, 0i), (ij, ji)$ of the original matrix Φ. As explained in the Appendix, in the eigenvalue representation of the partition function, the integral over all types of these ghosts produces the correct jacobian of the matrix model in the symmetry broken phase,

$$\Delta(\lambda) = \prod_i \prod_{\alpha<\beta} \left(\lambda_\alpha^{(i)} + \lambda_\beta^{(i)}\right)^2 \prod_i \prod_{\alpha,\beta} \left(\left(\lambda_\alpha^{(0)}\right)^2 - \left(\lambda_\beta^{(i)}\right)^2\right)^2 \prod_{i<j} \prod_{\alpha,\beta} \left(\left(\lambda_\alpha^{(i)}\right)^2 - \left(\lambda_\beta^{(j)}\right)^2\right)^2, \tag{3.8}$$

where $\lambda_\alpha^{(i)}$ stands for the eigenvalues in the i-th block. Integrating the kinetic terms of the ghosts then amounts to replacing the vacuum values $\lambda_\alpha^{(i)} = e_i, \lambda_\alpha^{(0)} = 0$ in the above expression. This will give

$$\int dBdC \mathrm{e}^{I_{\mathrm{kin}}(B,C;e)} = \prod_i (2e_i)^{M_i(M_i-1)} \prod_i (e_i)^{4M_0M_i} \prod_{i<j} (e_{ij})^{2M_iM_j}. \tag{3.9}$$

After inserting the g_s factors and ignoring the linear terms in M_0, M_i, the ghost contribution to $\mathcal{F}_0^{(\mathrm{np})}$ becomes

$$\sum_i S_i^2 \log(2e_i) + 4S_0 \sum_i S_i \log e_i + 2\sum_{i<j} S_iS_j \log(e_{ij}). \tag{3.10}$$

Finally, let us turn to the volume factor $(\mathrm{vol}\ G)^{-1}$ for the broken (matrix model) gauge group $G = \mathrm{SO}(2M_0) \times \mathrm{U}(M_1) \times \cdots \times \mathrm{U}(M_N)$. Using the asymptotic expansion of the volumes of the groups $\mathrm{SO}(2N)$ and $\mathrm{U}(N)$ in the large N limit (see [14, 8]), we can write it as

$$\begin{aligned}
\log(\mathrm{vol}\, G) &= \log(\mathrm{vol\ SO}(2M_0)) + \sum_i \log(\mathrm{vol\ U}(M_i)) \\
&= -M_0^2 \log M_0 + \left(\frac{3}{2} + \log\pi\right) M_0^2 + \mathcal{O}(M_0 \log M_0) + \\
&\quad + \sum_i \left[-\frac{1}{2}M_i^2 \log M_i + \left(\frac{3}{4} + \frac{1}{2}\log 2\pi\right) M_i^2 + \mathcal{O}(\log M_i)\right].
\end{aligned} \tag{3.11}$$

We have kept the next to leading order terms in the above expansion as they are crucial in cancellation of some numerical factors appearing later. The contribution of the volume factor to $\mathcal{F}_0^{(\mathrm{np})}$ thus becomes

$$M_0^2 \log M_0 + \frac{1}{2}\sum_i M_i^2 \log M_i - \left(\frac{3}{2} + \log\pi\right) M_0^2 - \left(\frac{3}{4} + \frac{1}{2}\log 2\pi\right) \sum_i M_i^2. \tag{3.12}$$

Summing the above three contributions and the linear terms $-\sum_i S_i W(e_i)$ coming from the vacuum value of $W(\Phi)$, we get the final result for the non-perturbative part of the free energy

$$\begin{aligned}\mathcal{F}_0^{(\mathrm{np})}(S) = & -\sum_i S_i W(e_i) + S_0^2 \log\left(\frac{S_0}{\alpha\hat\Lambda\hat\Delta_0}\right) + \frac{1}{2}\sum_i S_i^2 \log\left(\frac{S_i}{\alpha\hat\Lambda^2\hat\Delta_i}\right) + \\ & + 2S_0\sum_i S_i \log\left(\frac{e_i^2}{\hat\Lambda}\right) + 2\sum_{i<j} S_i S_j \log\left(\frac{e_{ij}}{\hat\Lambda}\right), \end{aligned} \tag{3.13}$$

where $\hat\Delta_0, \hat\Delta_i$ are defined as follows

$$\begin{aligned} e^{-3/2}\hat\Delta_0 &\equiv \Delta_0 = R_0\,, \\ e^{-3/2}\hat\Delta_i &\equiv \frac{\Delta_i}{2e_i^2} = R_i\,, \end{aligned} \tag{3.14}$$

and $\hat\Lambda$ is an arbitrary cut-off. Powers of $\hat\Lambda$ are inserted by hand in the above expression in a way to subtract the overall term $(S_0 + \sum_i S_i)^2 \log\hat\Lambda$ from $\mathcal{F}_0^{(\mathrm{np})}$. This corresponds to a freedom in choosing the scale of Φ in the original model. Indeed, by rescaling Φ as $\Phi \to \sqrt{\Lambda}\Phi$, the overall measure of the the $\mathrm{SO}(2M)$ matrix model scales as $d\Phi \to (\sqrt{\Lambda})^{2M^2-M} d\Phi$. This produces a change in the planar free energy as

$$\delta\mathcal{F}_0 = g_s^2(2M^2 - M)\log\sqrt{\Lambda}\,, \tag{3.15}$$

which in the 't Hooft limit (with $S = S_0 + \sum_i S_i$ a finite quantity) has precisely the same form $S^2\log\Lambda$ as we introduced in eq. (3.13).

As stated earlier, to calculate the effective couplings, we have to rewrite the free energy (3.13) by replacing S_i into $\widetilde{S}_i$ wherever (anti)symmetric fields are present,[3] i.e.,

$$\begin{aligned}\mathcal{F}_0^{(\mathrm{np})}(S) = & \sum_i \Bigg\{ -S_i W(e_i) + \frac{1}{2}S_i^2 \log\frac{S_i}{\alpha\hat\Lambda^2\Delta_i} + \frac{1}{2}S_i\widetilde{S}_i \log\frac{2e_i^2}{\hat\Lambda} + 2S_0\widetilde{S}_i \log\frac{e_i^2}{\hat\Lambda} + \\ & + \frac{1}{2}\sum_{j\neq i} S_i S_j \log\frac{(e_i - e_j)^2}{\hat\Lambda} + \frac{1}{2}\sum_{j\neq i} S_i\widetilde{S}_j \log\frac{(e_i + e_j)^2}{\hat\Lambda} \Bigg\} + \\ & + S_0^2 \log\frac{S_0}{\alpha\hat\Lambda\hat\Delta_0}\,. \end{aligned} \tag{3.16}$$

Notice that because of the symmetry breaking pattern in (2.5), a nonzero vev for S_0 does not make sense, and therefore we will eventually set it to zero. However, as we will see shortly, keeping S_0 will allow us to work out a simple rule relating the unoriented contributions $\mathcal{G}_0$ to the derivative of $\mathcal{F}_0$ with respect to S_0.

[3] These include $(B)C_{ii}^{1,3}, (B)C_{ij}^{1,3}$, which in the double line notation have index lines of the same orientation. But $\psi_{ii}^{0,2}$ or $(B)C_{ij}^{0,2}$ can be combined into matrices which have index lines of opposite directions.

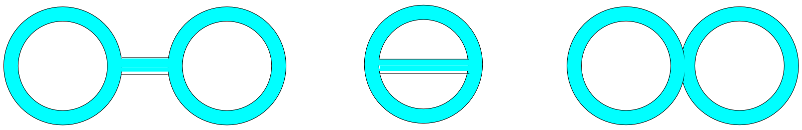

Figure 1: Two loops without ghosts.

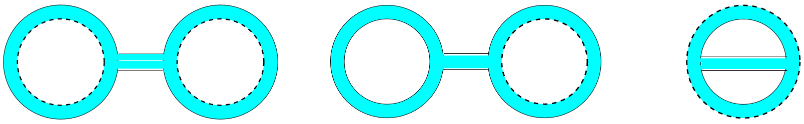

Figure 2: Two loops involving ghosts.

3.2 Two loop matrix model

Having obtained the propagators and the interaction terms up to the forth order around the vacuum (2.6) in section 2, we are now in a position to do the perturbative calculations of the free energy $\mathcal{F}$ in the planar limit and up to two vertices. Consider the two loops Feynmann diagrams in figure 1 and those including ghosts in figure 2.

The result of the two loop free energy calculation is

$$\begin{aligned}
\mathcal{F}_0^{(3)} = & \frac{1}{2}\sum_i Y_i \left(\frac{1}{2\Delta_i}\right) Y_i M_i - \sum_i \left(\frac{1}{6}+\frac{1}{2}\right)\left(\frac{1}{2\Delta_i}\right)^3 \gamma_{3,i}^2 M_i^3 - \\
& - 2\sum_i \left(\frac{1}{2}+\frac{1}{2}+1\right)\left(\frac{1}{2\Delta_i}\right)^2 \gamma_{4,i} M_i^3 - \sum_i \left(\frac{1}{2e_i}\right)^2 \left(\frac{1}{2\Delta_i}\right) M_i^2 \widetilde{M}_i - \\
& - \sum_i \sum_{j\neq i} \frac{1}{2\Delta_i}\left(\frac{1}{(e_i+e_j)^2} M_i^2 \widetilde{M}_j + \frac{1}{(e_i-e_j)^2} M_i^2 M_j\right) - \\
& - 4\sum_i \left(\frac{1}{e_i}\right)^2 \left(\frac{1}{\Delta_0}\right) M_i M_0^2 - 2\sum_i \left(\frac{1}{e_i}\right)^2 \left(\frac{1}{2\Delta_i}\right) M_i^2 M_0 + \\
& + \frac{1}{2}\left(\frac{1}{\Delta_0}\right) s_{N-1} (2M_0)^3 ,
\end{aligned} \tag{3.17}$$

where

$$Y_i = \left(\frac{2}{2e_i}\right)\widetilde{M}_i + \left(\frac{2}{2\Delta_i}\right)\imath\gamma_{3,i} M_i + \sum_{j\neq i}\left(\frac{2}{e_i-e_j} M_j + \frac{2}{e_i+e_j}\widetilde{M}_j\right) + \frac{4}{e_i} M_0 , \tag{3.18}$$

can be calculated from the tadpole graph.

Restoring the coefficients α and g_s, and taking $S_i = g_s M_i$ we find:

$$\begin{aligned}
\alpha\mathcal{F}_0^{(3)} = \sum_i \Bigg\{ & \left(-\frac{8}{3}\gamma_{3,i}^2 \frac{1}{(2\Delta_i)^3} - \frac{4}{(2\Delta_i)^2}\gamma_{4,i}\right) S_i^3 + \left(\frac{2\imath\gamma_{3,i}}{e_i(2\Delta_i)^2} - \frac{1}{8e_i^2\Delta_i}\right) S_i^2 \widetilde{S}_i + \\
& + \frac{1}{4e_i^2\Delta_i} S_i \widetilde{S}_i^2 - \sum_{j\neq i}\left(\frac{1}{2\Delta_i(e_i+e_j)^2} - \frac{4\imath\gamma_{3,i}}{(2\Delta_i)^2}\frac{1}{(e_i+e_j)}\right) S_i^2 \widetilde{S}_j -
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j\neq i}\left(\frac{1}{2\Delta_i(e_i-e_j)^2}-\frac{4\imath\gamma_{3,i}}{(2\Delta_i)^2}\frac{1}{(e_i-e_j)}\right)S_i^2S_j+\\
&+2\sum_{j\neq i}\left(\frac{1}{2\Delta_ie_i(e_i-e_j)}\widetilde{S}_iS_iS_j+\frac{1}{2\Delta_ie_i(e_i+e_j)}\widetilde{S}_iS_i\widetilde{S}_j\right)+\\
&+\sum_{j,k\neq i}\left(\frac{2}{2\Delta_i(e_i-e_j)(e_i-e_k)}S_iS_jS_k+\frac{2}{2\Delta_i(e_i+e_j)(e_i+e_k)}S_i\widetilde{S}_j\widetilde{S}_k\right)+\\
&+\sum_{j,k\neq i}\frac{4}{2\Delta_i(e_i-e_j)(e_i+e_k)}S_iS_j\widetilde{S}_k\Bigg\}+4\left(\frac{s_{N-1}}{\Delta_0}\right)(S_0)^3+\\
&+\sum_i\left\{\frac{4}{e_i^2\Delta_0}S_iS_0^2+\frac{1}{\Delta_ie_i^2}S_iS_0^2+\frac{\imath\gamma_{3,i}}{\Delta_i^2e_i}S_i^2S_0+\sum_{j\neq i}\frac{4}{\Delta_i(e_i^2-e_j^2)}S_0S_iS_j\right\}. \qquad (3.19)
\end{aligned}
$$

Note that $\Delta_0 \equiv s_N = \prod_i e_i^2$ and $\left(\frac{s_{N-1}}{s_N}\right) = \sum_i \frac{1}{e_i^2}$.

3.3 Unoriented planar contribution to the free energy

Here, we explicitly calculate the unoriented graphs contributions to the free energy. Notice that since ψ_{ii} (as well as B_{ii} and C_{ii}) matrices are antisymmetric, i.e., take value in the Lie algebra of $\mathrm{SO}(M_i)$, the corresponding propagators must accordingly be antisymmetrized

$$
\langle\psi_{\alpha\beta}\psi_{\gamma\delta}\rangle \sim \frac{1}{2}(\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\beta\delta}\delta_{\alpha\gamma})\,, \qquad (3.20)
$$

where $\alpha,\beta,\gamma\ldots = 1,\ldots,2M_i$ indicate the matrix indices. Therefore, unoriented planar graphs, i.e., graphs with the topology of sphere with a crosscap, must also be considered in the computation of the free energy in the planar limit. This will modify the expression for the effective superpotential to [1, 7]

$$
W_{\mathrm{eff}}^{\mathrm{pert}}(S) = -\sum_{i=0}^{N} N_i\frac{\partial\mathcal{F}_0}{\partial S_i} - \lambda\mathcal{G}_0\,, \qquad (3.21)
$$

Further, since in the case at hand the gauge group is broken to $\mathrm{U}(1)^N$ in the gauge theory side, we set $N_i = 1$ for $i \geq 0$.

We mentioned above that unoriented graphs come from the anti-symmetrization of the propagators for the antisymmetric matrices (more precisely from the second term on the r.h.s. of eq. (3.20)). In the present case, due to the decomposition in terms of the Pauli matrices and since $\sigma_0,\sigma_1,\sigma_3$ are symmetric while σ_2 is antisymmetric, the matrices ψ_{00}, ψ_{ii}^0, $B_{ii}^{1,3}$ and $C_{ii}^{1,3}$ become antisymmetric, whereas ψ_{ii}^2 is a symmetric one. Thus the propagator for ψ_{ii}^2 matrices will be,

$$
\langle\psi^2_{\alpha\beta}\psi^2_{\gamma\delta}\rangle \sim \frac{1}{2}(\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\beta\delta}\delta_{\alpha\gamma})\,. \qquad (3.22)
$$

As a result it can be seen that the contributions of ψ_{ii}^2 and ψ_{ii}^0 to the unoriented part in fact cancel each other. More interestingly, note that ψ_{ii}^2 and ψ_{ii}^0 can be put together to form a hermitean $M_i \times M_i$ matrix

$$
\psi_{ii} = \psi_{ii}^2 + i\psi_{ii}^0\,. \qquad (3.23)
$$

This is consistent with the symmetry breaking pattern in (2.7), and explains why these matrices do not have unoriented graphs.

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Next, let us write down the result of the calculations of the unoriented contributions to the free energy. Effectively, these are coming from a twist on ψ_{00}, $B_{ii}^{1,3}$, and $C_{ii}^{1,3}$ propagators. The unoriented contribution reads

$$\mathcal{G}_0 = -\sum_i S_i \frac{1}{2e_i}\frac{1}{\Delta_i} Y_i + \sum_i S_i^2 \left(\frac{1}{2e_i}\right)^2 \frac{1}{\Delta_i}\,. \tag{3.24}$$

It is easy to show that the above unoriented free energy can be derived by taking the derivative of the oriented part with respect to M_0 [7, 8], i.e.,

$$\mathcal{G}_0 = \frac{-1}{2}\frac{\partial}{\partial S_0}\mathcal{F}_0\,. \tag{3.25}$$

What we have done in this section is a nontrivial illustration of the above 'derivative rule' (3.25). This rule can be understood naively in some simpler examples. Putting a twist on a propagator reduces the number of index loops by one. This has to be done for each loop, and thus, starting with a graph of order S^n, we end up with a graph of order nS^{n-1} which is the derivative rule. It is interesting to note that not only the unoriented graphs of ψ_{00}, but also those constructed by a twist on $B_{ii}C_{ii}$ propagators, can both be rederived by differentiating $\mathcal{F}_0$ with respect to S_0.

4. Effective superpotential and $\mathcal{N}=2$ prepotential from matrix model

In the previous section, we derived the free energy of the SO$(2M)$ matrix model in the planar limit and up to two vertices. The prescription given by Dijkgraaf and Vafa [1] enables us now to write down the $\mathcal{N}=1$ effective superpotential $W_{\rm eff}(S)$, using (3.1). In the following subsections, we write down the details of these calculations.

4.1 Coupling constants from the matrix model

Let us start by computing $W_{\rm eff}(S)$ from (3.1) up to order $\mathcal{O}(S^3)$. After a little algebra we obtain

$$\begin{aligned} W_{\rm eff}(S) = & \sum_i \left(W(e_i) - S_i \log\frac{S_i}{\alpha\hat\Lambda^2\hat\Delta_i} - \frac{1}{2}S_i + S_i \log\frac{4e_i^2}{\hat\Lambda} - 2\sum_{j\neq i} S_j \log\frac{e_{ij}}{\hat\Lambda}\right) + \\ & + \sum_i \left(\frac{-\imath\gamma_{3,i}}{\Delta_i^3} - \frac{3\gamma_{4,i}}{\Delta_i^2} + \frac{3\imath\gamma_{3,i}}{2e_i\Delta_i^2} + \frac{3}{8e_i^2\Delta_i}\right) S_i^2 + \\ & + \sum_i\sum_{j\neq i}\left(-\frac{2(e_i^2+e_j^2)}{\Delta_i e_{ij}^2} + \frac{2\imath\gamma_{3,i}}{\Delta_i^2}\frac{e_i}{e_{ij}} + \frac{2}{\Delta_i e_{ij}}\right)(2S_iS_j + S_i^2) + \\ & + \sum_i\sum_{j\neq i}\sum_{k\neq i}\left(\frac{4e_i^2}{\Delta_i e_{ij} e_{ik}}\right)(S_iS_j + S_iS_k + S_jS_k) + 4\mathcal{G}_0\,. \end{aligned} \tag{4.1}$$

Upon extremizing $W_{\rm eff}(S)$, it is found that

$$\frac{\partial W_{\rm eff}}{\partial S_m} = \log\left(\frac{S_m}{\alpha\hat\Lambda^2\hat\Delta_m}\right) + \frac{3}{2} - \log\frac{4e_m^2}{\hat\Lambda} + 2\sum_{j\neq m}\log\frac{e_{jm}}{\hat\Lambda} + \frac{1}{\alpha}\sum_j A_{mj}S_j = 0\,, \tag{4.2}$$

where A_{mj} denote the coefficients of the quadratic part of $W_{\rm eff}(S)$ in (4.1).

For small $\hat{\Lambda}$, we can solve the equation (4.2) by iteration to find the roots $S_m = \langle S_m \rangle$. The result up to the second order is given by

$$\langle S_m \rangle = \alpha \tilde{\Lambda} \frac{4e_m^2}{R_m} - \alpha \tilde{\Lambda}^2 \sum_j A_{mj} \frac{4e_m^2}{R_m} \frac{4e_j^2}{R_j} \,, \tag{4.3}$$

where we have defined the new cut-off $\tilde{\Lambda}$ as

$$\tilde{\Lambda} \equiv \hat{\Lambda}^{2(N+n_0)} \exp\left(2i\pi\tau_0\right) \equiv \Lambda^{2(N+n_0)} \,. \tag{4.4}$$

The real gauge theory cut-off Λ is the one defined by the last equality.

We note that the perturbative and d-instanton parts of τ_{mn} in the above decomposition come from $\mathcal{F}_0^{(\mathrm{np})}$ and $\mathcal{F}_0^{(d+2)}$ terms in the matrix model side, respectively. By differentiating $\mathcal{F}_0^{(\mathrm{np})}(S)$ and $\mathcal{F}_0^{(3)}(S)$ according to the rules (3.3), (3.4), at the point $S_i = \langle S_i \rangle$, one can find $\tau_{mn}^{(\mathrm{pert})}$ and $\tau_{mn}^{(1)}$ in terms of e_i's,

$$\begin{aligned}
2\pi\imath\tau_{mn}^{(\mathrm{pert})} &= \delta_{mn}\left\{-2\sum_i \log\frac{e_{im}}{\hat{\Lambda}} - \log\frac{4e_m^2}{\hat{\Lambda}}\right\} + (1-\delta_{mn})\log\frac{(e_n-e_m)^2}{(e_n+e_m)^2}\,, && (4.5)\\
2\pi\imath\tau_{mn}^{(1)} &= \delta_{mn}\Bigg\{\left(\frac{-2\gamma_{3,m}^2}{\Delta_m^3} - \frac{6\gamma_{4,m}}{\Delta_m^2} - \frac{\imath\gamma_{3,m}}{e_m\Delta_m^2} - \frac{1}{4e_m^2\Delta_m}\right)\frac{4e_m^2}{R_m} + \\
&\quad + \sum_{j\neq m}\frac{4e_j^2}{R_j}\left(-\frac{2(e_m^2+e_j^2)}{\Delta_m e_{mj}^2} + \frac{4\imath\gamma_{3,m}}{\Delta_m^2}\frac{e_m}{e_{mj}} + \frac{8e_m^2}{\Delta_j e_{mj}}\right) - \\
&\quad - \sum_j A_{mj}\frac{4e_m^2}{R_m}\frac{4e_j^2}{R_j}\Bigg\} + (1-\delta_{mn})\times \\
&\quad \times\Bigg\{\frac{16e_m^2 e_n}{\Delta_m R_m}\left(\frac{-e_m}{e_{mn}^2} + \frac{\imath\gamma_{3,m}}{\Delta_m e_{mn}}\right) + \frac{16e_n^2 e_m}{\Delta_n R_n}\left(\frac{-e_n}{e_{mn}^2} + \frac{\imath\gamma_{3,n}}{\Delta_n e_{nm}}\right) + \\
&\quad + \sum_{j\neq m}\frac{8e_m e_n}{\Delta_m e_{mn} e_{mj}}\frac{4e_j^2}{R_j} + \sum_{j\neq n}\frac{8e_m e_n}{\Delta_n e_{nm} e_{nj}}\frac{4e_j^2}{R_j} + \sum_{j\neq m,n}\frac{8e_m e_n}{\Delta_j e_{jm} e_{jn}}\frac{4e_j^2}{R_j}\Bigg\}\,. && (4.6)
\end{aligned}$$

As expected, these quantities turn out to be independent of the parameter α. Therefore, the coupling constants of the unbroken U(1) factors of the $\mathcal{N} = 2$ gauge theory are given by the mn components of the above equation.

4.2 Computation of the periods within the Matrix Model

In ref. [6] a method was proposed for the computation of the periods a_i of the Seiberg-Witten curve. The method is in fact based on a purely perturbative calculation of the planar tadpole diagrams within the matrix model with no reference to the actual form of the Seiberg-Witten curve or differential. Here, within the same framework as in [6], we use a rather different method based on differentiating with respect to the variation of the potential of the matrix model by linear source terms. To be specific, let us consider the original matrix model with linear source terms of the form $-\sum_i \epsilon_i \operatorname{Tr}(\phi_{ii}^2)$, with ϵ_i infinitesimal parameters. The planar free energy of this modified matrix model is given by the following equation

$$\exp\left(\frac{1}{g_s^2}\mathcal{F}_0'\right) = \frac{1}{\mathrm{vol\,G}}\int d\Phi \exp\left(-\frac{1}{g_s}\left(W(\Phi) - \sum_i \epsilon_i \operatorname{Tr}(\phi_i)\right)\right)_{\text{planar}}, \tag{4.7}$$

where we have put[4] $\phi_i \equiv \phi_{ii}^2$. After all, this implies a simple relation between the planar tadpole diagrams given by $\langle \operatorname{Tr}(\phi_i)\rangle_0$ and the free energy as[5]

$$\langle \operatorname{Tr}(\phi_i)\rangle_0 = \frac{1}{g_s}\frac{\delta \mathcal{F}_0}{\delta \epsilon_i}. \tag{4.8}$$

Adding the source terms amounts to replacing the block superpotentials by

$$\operatorname{Tr} w(\phi_i) \to \operatorname{Tr} \tilde{w}_i(\phi_i) \equiv \operatorname{Tr} w(\phi_i) - \epsilon_i \operatorname{Tr}(\phi_i), \tag{4.9}$$

in which

$$\begin{aligned} W(\phi) &\equiv \sum_i \operatorname{Tr} w(\phi_i) \\ \tilde{w}_i(x) &\equiv w(x) - \epsilon_i x. \end{aligned} \tag{4.10}$$

This modification clearly changes the vacuum of the matrix model. The true shift in the vacuum can be easily obtained by going to the eigenvalue representation of the matrix model. In this representation the vacuum values of $\lambda^{(i)}$'s in different blocks are determined by extremizing the associated superpotentials, that is for the ii block by the equation

$$w'(x) = \epsilon_i. \tag{4.11}$$

This change in the vacuum causes the zero point energies, the couplings, and the propagators of the original matrix model shift according to the following relations[6]

$$\begin{aligned} e_i &\to \tilde{e}_i = e_i + \frac{\epsilon_i}{w''(e_i)}, \\ w(e_i) &\to \tilde{w}_i(\tilde{e}_i) = w(e_i) - \epsilon_i e_i, \\ w^{(l)}(e_i) &\to \tilde{w}_i^{(l)}(\tilde{e}_i) = w^{(l)}(e_i) + \epsilon_i \frac{w^{(l+1)}(e_i)}{w''(e_i)}, \qquad l \geq 2. \end{aligned} \tag{4.12}$$

It is important to note that, although the quantities $w^{(l)}(e_i)$ are explicit functions of all e_i's, in the above procedure, we have replaced only e_i in the argument of $w^{(l)}(x)$, holding all its (e_i-dependent) coefficients fixed. Since the planar free energy $\mathcal{F}_0$ is in general a

[4] Note that we do not need to consider a source for ϕ_{00} block, since it is an antisymmetric matrix and has $\operatorname{Tr}(\phi_{00}) = 0$, corresponding to $a_0 = 0$. Also ϕ_{ii}^0 has zero trace.

[5] For the precise definition of the operators $\frac{\delta}{\delta \epsilon_i}$ see below.

[6] The $l = 2$ choice in the last line of these equations corresponds to a modification of the propagators of ψ_i, while the $l > 2$ choice gives the changes in their vertex factors.

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function of S_I and the parameters $e_i, w(e_i), w^{(l)}(e_i)$, the above discussion indicates that the addition of the source terms has the net effect of changing $\mathcal{F}_0$ as follows

$$\mathcal{F}_0\left(S_I, e_i, w(e_i), w^{(l)}(e_i)\right) \to \mathcal{F}_0' \equiv \mathcal{F}_0\left(S_I, \tilde{e}_i, \tilde{w}_i(\tilde{e}_i), \tilde{w}_i^{(l)}(\tilde{e}_i)\right). \tag{4.13}$$

In particular, this shows that the differential operator $\frac{\delta}{\delta\epsilon_i}$ must be defined precisely as follows

$$\frac{\delta}{\delta\epsilon_i} \equiv -e_i\frac{\partial}{\partial w(e_i)} + \frac{1}{w''(e_i)}\left(\frac{\partial}{\partial e_i} + \sum_{l\geq 2} w^{(l+1)}(e_i)\frac{\partial}{\partial w^{(l)}(e_i)}\right). \tag{4.14}$$

Now, we turn to the calculation of the Seiberg-Witten periods. By the same line of reasoning as in [6], we define the periods a_i in the matrix model using the following equation (It can also be computed in terms of the planar tadpole diagrams)

$$a_i = g_s\sum_{K=0}^{N} n_K\left(\frac{\partial}{\partial S_K}\langle \mathrm{Tr}(\phi_i)\rangle_0\right)_{\langle S\rangle}, \tag{4.15}$$

where n_I are defined as follows:

$$\begin{aligned} n_i &= 1\,, \qquad i = 1,\ldots,N \\ n_0 &= \frac{N_0}{2} - 1\,. \end{aligned} \tag{4.16}$$

Note that for SO($2N$), $n_0 = -1$, and for SO($2N+1$), $n_0 = -1/2$.

Upon expanding around the vacuum, $\phi_i = e_i + \psi_i$, one sees that $\langle \mathrm{Tr}(\phi_i)\rangle_0 = e_i M_i + \langle \mathrm{Tr}(\psi_i)\rangle$. eq. (4.15) now implies the expected expansion of a_i as $a_i = e_i + \mathcal{O}(\langle S\rangle)$. Using the relation $\langle \mathrm{Tr}(\phi_i)\rangle_0 = \frac{1}{g_s}\frac{\delta\mathcal{F}_0}{\delta\epsilon_i}$ for the tadpole, we can rewrite a_i as

$$a_i = \left(\frac{\delta}{\delta\epsilon_i}\sum_{K=0}^{N} n_K\frac{\partial\mathcal{F}_0}{\partial S_K}\right)_{\langle S\rangle}. \tag{4.17}$$

Noticing the general formula for W_{eff} in terms of $\mathcal{F}_0$, we are led to the final general formula for a_i

$$a_i = \left(\frac{\delta W_{\mathrm{eff}}}{\delta\epsilon_i}\right)_{\langle S\rangle} = \frac{\delta}{\delta\epsilon_i}W_{\mathrm{eff}}(\langle S\rangle)\,, \tag{4.18}$$

where in the last step, we have used the fact that $\partial W_{\mathrm{eff}}/\partial S = 0$ at $\langle S\rangle$. In using the above formula, we should be careful to use the full expression of $W_{\mathrm{eff}}(\langle S\rangle)$ in terms of e_i and $w^{(l)}(e_i)$ without using the explicit forms of $w^{(l)}(e_i)$ in terms of e_i's.

In the case of our interest, $W_{\mathrm{eff}}(\langle S\rangle)$ can be expressed as functions of e_i, $w''(e_i)$, $w^{(3)}(e_i)$, and $w^{(4)}(e_i)$. Thus our formula (4.18) up to the order $\tilde{\Lambda}$ is simplified to

$$a_i = e_i - \frac{\tilde{\Lambda}}{w''(e_i)}\left(\frac{\partial}{\partial e_i} + w^{(3)}(e_i)\frac{\partial}{\partial w''(e_i)}\right)W_{\mathrm{eff}}^{(1)} + \mathcal{O}(\tilde{\Lambda}^2)\,, \tag{4.19}$$

where $W_{\mathrm{eff}}^{(1)}$ is the first order term of W_{eff}. Thus for a_i we find

$$a_i = e_i - \frac{2\tilde{\Lambda}}{w''(e_i)}\left((2-4n_0)\frac{(e_i)^{1-4n_0}}{w''(e_i)} - w^{(3)}(e_i)\frac{(e_i)^{2-4n_0}}{(w''(e_i))^2}\right) + \mathcal{O}(\tilde{\Lambda}^2)\,. \tag{4.20}$$

Finally for SO$(2N+1)$ case, we find e_i's in terms of a_i's as follows:

$$e_i = a_i + \tilde{\Lambda} f_i + \mathcal{O}(\tilde{\Lambda}^2)\,, \tag{4.21}$$

where

$$\begin{aligned} f_i &= \frac{1}{4a_i R_i^2}\left(1 - 4a_i^2 \sum_{j\neq i} \frac{1}{a_{ij}}\right) \\ R_i &= \prod_{k\neq i}(a_k - a_i)\,. \end{aligned} \tag{4.22}$$

For determining the prepotential $\mathcal{F}(a)$, we need to express $\tau_{ij}(e)$ in terms of a_i instead of e_i using (4.21). Thus we obtain,

$$\tau_{ij}(a) = \tau_{ij}^{(\mathrm{pert})}(a) + \tilde{\Lambda}\left(\tau_{ij}^{(1)}(a) + \sum_k \frac{\partial \tau_{ij}^{(\mathrm{pert})}(a)}{\partial a_k} f_k(a)\right) + \mathcal{O}(\tilde{\Lambda}^2)\,. \tag{4.23}$$

Here, $\tau_{ij}^{(\mathrm{pert})}(a)$ and $\tau_{ij}^{(1)}(a)$ are given by replacing $e_i \to a_i$ in (4.5) and (4.6). The final results are as follows,

$$\begin{aligned} 2\pi\imath\tau_{mn}^{(\mathrm{pert})} &= \delta_{mn}\left\{-2\sum_i \log\frac{a_{im}}{\tilde{\Lambda}} - \log\frac{4a_m^2}{\hat{\Lambda}}\right\} + (1-\delta_{mn})\log\frac{(a_n-a_m)^2}{(a_n+a_m)^2}\,, \\ 2\pi\imath\tau_{mn}^{(1)} &= \delta_{mn}\sum_{j\neq m}\left(\frac{-4}{R_m^2 a_{mj}} + \frac{24a_m^2}{R_m^2 a_{mj}^2} + \frac{24a_m^2}{R_j^2 a_{mj}^2} - \frac{-4}{R_j^2 a_{mj}} + \frac{16a_m^2}{R_m^2}\sum_{k\neq m,j}\frac{1}{a_{mj}a_{mk}}\right) + \\ &\quad + (1-\delta_{mn})\left\{\frac{-8a_m a_n}{a_{mn}^2}\left(\frac{1}{R_m^2} + \frac{1}{R_n^2}\right) + 16a_m a_n \sum_{j\neq m,n}\frac{1}{R_j^2 a_{nj} a_{mj}} - \right. \\ &\quad \left. - \frac{16a_m a_n}{a_{nm}}\left(\frac{1}{R_n^2}\sum_{j\neq n}\frac{1}{a_{nj}} - \frac{1}{R_m^2}\sum_{j\neq m}\frac{1}{a_{mj}}\right)\right\}. \end{aligned} \tag{4.24}$$

It is easy to show that the above expression for $\tau_{mn}(a)$ can be integrated to give the following prepotential $\mathcal{F}(a)$, up to the one-instanton correction:

$$\begin{aligned} \mathcal{F}(a)^{(\mathrm{pert})} &= \frac{\imath}{4\pi}\left\{\sum_l \sum_{k\neq l}\left((a_k+a_l)^2 \log\frac{(a_k+a_l)^2}{\hat{\Lambda}} + (a_k-a_l)^2\log\frac{(a_k-a_l)^2}{\hat{\Lambda}}\right) + \right. \\ &\quad \left. + 2\sum_k a_k^2 \log\frac{a_k^2}{\hat{\Lambda}}\right\}, \\ \mathcal{F}(a)^{(1)} &= \frac{\tilde{\Lambda}}{16\pi\imath}\sum_k \prod_{l\neq k}\frac{1}{(a_k^2 - a_l^2)^2}\,, \end{aligned} \tag{4.25}$$

which is in agreement with the known results in the $\mathcal{N}=2$ theory [16].

5. Conclusion

We studied the $\mathcal{N}=2$ theory with the gauge group SO using the Dijgkraaf-Vafa proposal of Matrix Model approach to the $\mathcal{N}=1$ SYM theories. This was done by adding a superpotential to the $\mathcal{N}=2$ theory which broke it to $\mathcal{N}=1$, then using the corresponding matrix model, we computed the effective action for $\mathcal{N}=1$ gauge theory, with a nontrivial vacuum breaking the group into its maximal abelian subgroup. We chose this vacuum as we were interested in finding the $\mathcal{N}=2$ prepotential in the Coulomb branch. For this reason, and to derive the $\mathcal{N}=2$ effective couplings, we finally turned off the superpotential by sending its coefficient α to zero. As expected, the coupling constants τ_{ij} were independent of α and thus were identified with the $\mathcal{N}=2$ effective U(1) couplings. At the end, τ_{ij} were integrated to find out the prepotential of $\mathcal{N}=2$ theory.

In the calculation of the effective action, we carefully considered the unoriented graphs of the anti-symmetric matrices, and observed that their contributions can be rederived from the derivative of planar graphs with respect to the supergluball field, S_0. This provided an interesting and nontrivial example for the 'derivative rule'.

We also computed the periods of $\mathcal{N}=2$ theory by adding a source term to the matrix model action. This is equivalent to computing the tadpole graphs. However, the calculation of periods we did is general enough to be used in similar matrix models.

The extension to $\mathrm{SP}(2N)$ and $\mathrm{SO}(2N)$ gauge groups is straightforward. For $\mathrm{SO}(2N)$ group, the calculation steps are very much similar to that of $\mathrm{SO}(2N+1)$, though, a complication may arise due to the presence of $\mathrm{Pfaff}(\phi)$ in the superpotential.

Note added. During the course of this investigation the paper [17] appeared which considers $\mathcal{N}=1$ SO / SP gauge theories. They have derived the effective action.

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A. Faddeev-Popov ghost action

In this appendix, we will show that the Faddeev-Popov ghost action needed to fix the gauge to $\Phi_{ij}=0$, $i\neq j$ is the one given by (2.13). To begin with, let us first diagonalize the matrix Φ by an orthogonal $\mathrm{SO}(2M)$ transformation, and call the eigenvalues λ_I,

$$\Phi=\mathrm{diag}(\lambda_1 i\sigma_2,\ldots,\lambda_M i\sigma_2)\,. \tag{A.1}$$

The superpotential (2.1) is thus

$$W(\lambda)=2\alpha\sum_{l=0}^{N}\sum_{I=1}^{M}\frac{s_{N-l}(e^2)}{2l+2}\lambda_I^{2l+2}\,. \tag{A.2}$$

Further, if we define

$$\phi_i \equiv \mathrm{diag}(\lambda_1(i\sigma_2), \lambda_2(i\sigma_2), \ldots, \lambda_{M_i}(i\sigma_2)) \,, \tag{A.3}$$

then

$$\Phi = \mathrm{diag}(\phi_1, \ldots, \phi_N) \,. \tag{A.4}$$

The superpotential (A.2) can now be written as

$$W(\Phi) = \alpha \sum_{l=0}^{N} \sum_{I=1}^{N} \frac{s_{N-l}(e^2)}{2l+2} \, \mathrm{tr}\, \Phi_I^{2l+2} \,. \tag{A.5}$$

In diagonalizing the Φ matrix, one also has to take into account the Vandermonde determinant, which appears in the measure as the jacobian of the transformation. For the group $\mathrm{SO}(2M)$, this determinant reads

$$\Delta = \prod_{I \neq J}^{M} (\lambda_I^2 - \lambda_J^2) = \Delta^{(1)} \cdot \Delta^{(2)} \,, \tag{A.6}$$

where

$$\begin{aligned} \Delta^{(1)} &= \prod_{I_1 \neq J_1}^{M_1} (\lambda_{I_1}^2 - \lambda_{J_1}^2) \prod_{I_2 \neq J_2}^{M_2} (\lambda_{I_2}^2 - \lambda_{J_2}^2) \ldots \prod_{I_N \neq J_N}^{M_N} (\lambda_{I_N}^2 - \lambda_{J_N}^2) \\ \Delta^{(2)} &= \prod_{I_i, J_j (i \neq j)} \left(\lambda_{I_i}^2 - \lambda_{J_j}^2 \right) . \end{aligned} \tag{A.7}$$

Let us now write the second part of the Vandermonde determinant $\Delta^{(2)}$ as an integral over ghosts. First note that for a fixed λ_1 and λ_2 we have

$$(\lambda_1^2 - \lambda_2^2)^2 = \int dB_{21} dC_{12} \exp\left(B_{21}^{\alpha\beta} \lambda_1 (i\sigma_2)_{\beta\alpha} C_{12}^{\alpha\beta} + C_{12}^{\alpha\beta} \lambda_2 (i\sigma_2)_{\beta\alpha} B_{21}^{\alpha\beta} \right) , \tag{A.8}$$

where $\alpha, \beta = 1, 2$. Therefore

$$\begin{aligned} \prod_{I_1, J_2}^{M_1, M_2} (\lambda_{I_1}^2 - \lambda_{J_2}^2)^2 &= \int \prod_{I_1, J_2} dB_{J_2 I_1} dC_{I_1 J_2} \times \\ &\times \exp\left(\sum_{I_1, J_2}^{M_1, M_2} B_{J_2 I_1}^{\alpha\beta} \lambda_{I_1} (i\sigma_2)_{\beta\alpha} C_{I_1 J_2}^{\alpha\beta} + C_{I_1 J_2}^{\alpha\beta} \lambda_{J_2} (i\sigma_2)_{\beta\alpha} B_{J_2 I_1}^{\alpha\beta} \right) . \end{aligned} \tag{A.9}$$

Using definition (A.3), this can be written as

$$\prod_{I_1, J_2} (\lambda_{I_1}^2 - \lambda_{J_2}^2)^2 = \int dB_{21} dC_{12} \exp\left(\mathrm{tr}_2 (B_{21} \phi_1 C_{12}) + \mathrm{tr}_1 (C_{12} \phi_2 B_{21}) \right) , \tag{A.10}$$

where the subindex i indicates the trace is over $2M_i \times 2M_i$ matrices. It is also understood that B_{ji} and C_{ij} are $2M_j \times 2M_i$ and $2M_i \times 2M_j$ matrices, respectively. The Vandermonde determinant $\Delta^{(2)}$ now reads

$$\prod_{I_i, J_j} (\lambda_{I_i}^2 - \lambda_{J_j}^2)^2 = \int \prod_{i<j} dB_{ji} dC_{ij} \exp\left(\sum_{i<j} \mathrm{tr}_j (B_{ji} \phi_i C_{ij}) + \mathrm{tr}_i (C_{ij} \phi_j B_{ji}) \right) . \tag{A.11}$$

Therefore, the partition function turns out to be

$$Z = \int d\Phi dBdC \exp\left(\alpha \sum_{l,i} \frac{s_{N-l}(e^2)}{2l+2} \operatorname{tr} \phi_i^{2l+2} + \sum_{i<j} \operatorname{tr}_j(B_{ji}\phi_i C_{ij}) + \operatorname{tr}_i(C_{ij}\phi_j B_{ji})\right), \tag{A.12}$$

where the measure is

$$d\Phi dBdC = \prod_I d\lambda_I \prod_{I_1 \neq J_1}^{M_1} (\lambda_{I_1}^2 - \lambda_{J_1}^2) \prod_{I_2 \neq J_2}^{M_2} (\lambda_{I_2}^2 - \lambda_{J_2}^2) \cdots \prod_{I_N \neq J_N}^{M_N} (\lambda_{I_N}^2 - \lambda_{J_N}^2) \prod_{i<j} dB_{ji} dC_{ij} . \tag{A.13}$$

With the Vandermonde determinant $\Delta^{(1)}$ in the measure (A.13), one cannot go very far in perturbation theory. However, $\Delta^{(1)}$ can be re-absorbed in the action; simply drop the determinant, and in effect change the σ_2-diagonal ϕ_i matrices into some $2M_i \times 2M_i$ matrices ϕ_{ii} with λ_{I_i}'s as their eigenvalues. At the end, the partition function will be

$$Z = \int \prod_i d\phi_{ii} \prod_{i<j} dB_{ji} dC_{ij} \exp\left(W(\Phi) + \sum_{i<j}^{N} \operatorname{tr}_j(B_{ji}\phi_{ii} C_{ij}) + \operatorname{tr}_i(C_{ij}\phi_{jj} B_{ji})\right). \tag{A.14}$$

Noticing that $B_{ji} = -B_{ij}^T$ and $C_{ji} = -C_{ij}^T$ (as B and C are $\mathrm{SO}(2M)$ Lie algebra valued), the ghost action can be written as

$$S_{\mathrm{gh}} = \frac{1}{2} B[\Phi, C] , \tag{A.15}$$

which is the same action written in (2.13).

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